



PHYS 102 – General Physics II Final Exam

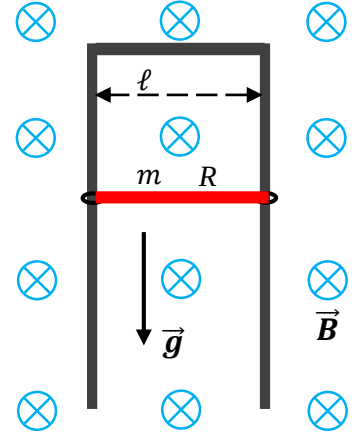
Duration: 120 minutes

Saturday, 28 December 2024; 15:00

1. The figure illustrates a horizontal metal wire of length ℓ , mass m and resistance R which slides without friction down two vertical resistanceless rails. Over the entire region, there is a constant uniform magnetic field of magnitude B , with direction into the plane of the paper. At time $t = 0$, the wire starts to fall under the action of the constant gravitational force.

(a) (10 Pts.) Find the terminal velocity (i.e. $\lim_{t \rightarrow \infty} v(t)$) of the wire.

(b) (15 Pts.) Find the velocity of the wire as a function of time.



Solution:

(a) Taking the downward direction as positive, net force on the horizontal wire is $F = mg - F_B$, where the magnitude of the magnetic force is given by

$$F_B = I_{\text{ind}} B \ell, \quad I_{\text{ind}} = \frac{\mathcal{E}_{\text{ind}}}{R} = \frac{B \ell v}{R} \rightarrow F_B = \left(\frac{B^2 \ell^2}{R} \right) v.$$

Therefore, Newton's second law is written as

$$m \frac{dv}{dt} = mg - \left(\frac{B^2 \ell^2}{R} \right) v \rightarrow \frac{dv}{dt} = g - \left(\frac{B^2 \ell^2}{mR} \right) v.$$

In the limiting case the terminal speed is constant, so

$$v_T = \left(\frac{mgR}{B^2 \ell^2} \right).$$

(b) Before the terminal speed is reached, we have

$$\frac{dv}{dt} = g - \left(\frac{B^2 \ell^2}{mR} \right) v \rightarrow \int \frac{dv}{v - v_T} = - \left(\frac{B^2 \ell^2}{mR} \right) \int dt \rightarrow \ln(v - v_T) = - \left(\frac{B^2 \ell^2}{mR} \right) t + C,$$

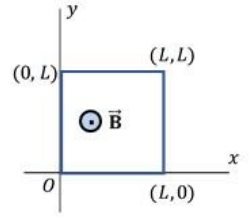
where C is an integration constant. Hence, we have

$$v - v_T = e^C e^{-\left(\frac{B^2 \ell^2}{mR} \right) t}.$$

The wire is at rest when $t = 0$, meaning that $v(0) = 0$ and $e^C = -v_T$. So

$$v(t) = v_T \left(1 - e^{-\left(\frac{B^2 \ell^2}{mR} \right) t} \right).$$

2. A square wire loop of side L and total resistance R is placed on the xy plane so that its corners are at $(0,0)$, $(0,L)$, (L,L) , and $(L,0)$. A non-uniform magnetic field that is increasing with time given as $\vec{B}(x, y, t) = (A x t) \hat{k}$, ($A > 0$), is applied from $t = 0$.



- (a) (4 Pts.) Is the current through the loop clockwise or counterclockwise?
- (b) (7 Pts.) Calculate the flux through the loop at time t .
- (c) (7 Pts.) Calculate the current through the loop at time t .
- (d) (7 Pts.) Calculate the net force on the current loop at time t . Clearly specify its magnitude and direction.

Solution:

(a) Magnitude of the magnetic field in the region is increasing in time. According to Lenz's law, the induced current should create a magnetic field in the opposite direction. Hence it must be **clockwise**.

(b) the magnitude of the magnetic field also depends on the spatial variable x . therefore, to calculate the flux of the magnetic field through the square loop we need to integrate. The flux through the shaded infinitesimal area is

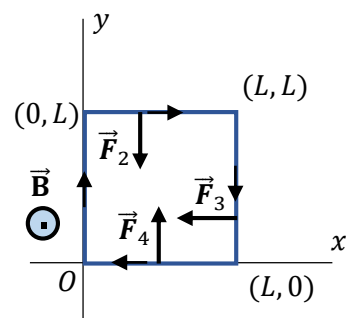
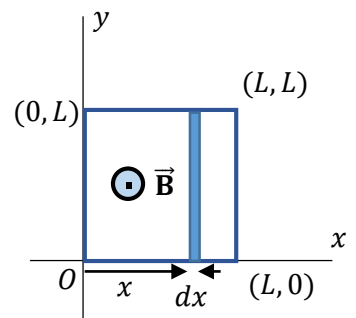
$$d\Phi_B = B L dx \rightarrow d\Phi_B = ALt x dx$$

$$\Phi_B = \int_0^L d\Phi_B \rightarrow \Phi_B = ALt \int_0^L x dx \rightarrow \Phi_B = \frac{1}{2} AL^3 t.$$

(c)

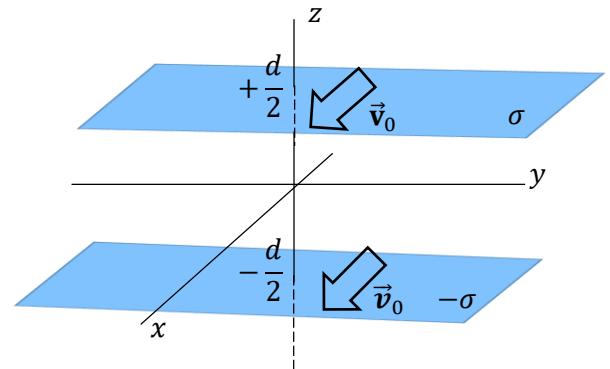
$$\mathcal{E}_{\text{ind}} = \frac{d\Phi_B}{dt} = \frac{1}{2} AL^3, \quad I_{\text{ind}} = \frac{\mathcal{E}_{\text{ind}}}{R} \rightarrow I_{\text{ind}} = \frac{AL^3}{2R}.$$

(d) Since the induced current is clockwise, forces on the loop are in the directions shown. Since the magnetic field is independent of the coordinate y , forces $\vec{F}_2$ and $\vec{F}_4$ on the top and the bottom sections of the loop cancel each other. Force on the segment at $x = 0$ is zero because the magnetic field is zero along the y -axis. Hence, the net force on the loop is $\vec{F}_3$, which is in the negative x direction.



$$\vec{F}_3 = I_{\text{ind}} L B(L) (-\hat{i}) \rightarrow \vec{F}_3 = \left(\frac{AL^3}{2R} \right) L (AL t) (-\hat{i}) \rightarrow \vec{F}_3 = \left(\frac{A^2 L^5 t}{2R} \right) (-\hat{i})$$

3. Two infinite parallel plates are placed at $z = +\frac{d}{2}$ and $z = -\frac{d}{2}$ with planes parallel to the xy plane. The upper plate carries a uniform surface charge of σ ($\sigma > 0$) and moves with speed v_0 in the positive x direction. The lower plate carries an opposite charge density of $-\sigma$ and moves with the same speed in the same direction. Calculate the following quantities: (Indicate the direction as well as the magnitude if the quantity is a vector, and write your answer in the boxes provided.)



- (a) (3 Pts.) the electric field at $z = 0$ between the plates;
- (b) (3 Pts.) the electric field above the upper plate;
- (c) (3 Pts.) the magnetic field at $z=0$, between the plates;
- (d) (3 Pts.) the magnetic field below the lower plate;
- (e) (3 Pts.) the energy density per unit volume for the region between the plates;
- (f) (4 Pts.) the Poynting vector between the plates;
- (g) (3 Pts.) the pressure (net force per unit area) on the upper plate.
- (h) (3 Pts.) What should v_0 be so that the pressure on the upper plate is 0?

Solution:

$$(a) \vec{E} = \frac{\sigma}{\epsilon_0} (-\hat{k})$$

By Gauss's law.

$$(b) \vec{E} = 0$$

By Gauss's law.

$$(c) \vec{B} = \sigma\mu_0 v_0 (\hat{j})$$

By Ampère's law.

$$(d) \vec{B} = 0$$

By Ampère's law.

$$(e) u = \frac{\sigma^2}{2} \left(\frac{1}{\epsilon_0} + \mu_0 v_0^2 \right)$$

$$u = u_E + u_B = \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 = \frac{\sigma^2}{2\epsilon_0} + \frac{1}{2\mu_0} (\sigma^2 \mu_0^2 v_0^2)$$

$$(f) \vec{S} = \frac{\sigma^2 v_0}{\epsilon_0} (\hat{i})$$

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} \frac{\sigma}{\epsilon_0} (-\hat{k}) \times \sigma\mu_0 v_0 (\hat{j}) = \frac{\sigma^2 v_0}{\epsilon_0} (\hat{i})$$

$$(g) p = \frac{\sigma^2}{2\epsilon_0} - \frac{\sigma^2 \mu_0 v_0^2}{2}$$

$$p = \frac{F}{A} = \frac{F_E + F_B}{A} = \frac{1}{2} \sigma E - \frac{1}{2} JB$$

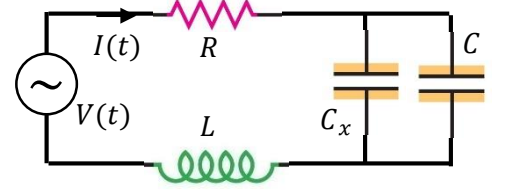
$$(h) v_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

4. An AC voltage source $V(t) = V_0 \sin(t/\sqrt{LC})$ is connected to a resistance R , an inductor L , and two parallel capacitors as shown in the figure. One of the capacitors has capacitance C while the second one is a variable capacitor which has capacitance C_x , which can be adjusted.

(a) (9 Pts.) Calculate the rms value of the current through the resistor.

(b) (8 Pts.) For which value of C_x is the average power dissipated in the circuit greatest?

(c) (8 Pts.) For which value of C_x is the time average value of the energy stored in the variable capacitor C_x maximum?



Solution: The circuit is an LRC series circuit with the equivalent capacitance $C_{eq} = C + C_x$. The current is the same for all three components R , C_{eq} , and L . Voltage on the resistance is in phase with the current, voltage on the equivalent capacitance lags the current by $\pi/2$, and the voltage on the inductor leads the current by $\pi/2$. Hence, we have the following phasor diagram. The impedance of the circuit is found as

$$V_0 = I_0 \sqrt{R^2 + (X_L - X_C)^2} \rightarrow Z = \frac{V_0}{I_0} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C_{eq}}\right)^2}.$$

Since $\omega = 1/\sqrt{LC}$, and $C_{eq} = C + C_x$, We have

$$Z = \frac{V_0}{I_0} = \sqrt{R^2 + \omega^2 \left(L - \frac{1}{\omega^2 C_{eq}}\right)^2} \rightarrow Z = \sqrt{R^2 + \frac{L}{C} \left(\frac{C_x}{C + C_x}\right)^2}$$

(a)

$$I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{V_0}{\sqrt{2}Z} \rightarrow I_{rms} = \frac{V_0}{\sqrt{2}} \frac{1}{\sqrt{R^2 + \frac{L}{C} \left(\frac{C_x}{C + C_x}\right)^2}}.$$

(b) Power is only dissipated in the resistor, and the average power dissipated is $P_{av} = RI_{rms}^2$. Since

$$P_{av} = \frac{1}{2} \frac{RV_0^2}{R^2 + \frac{L}{C} \left(\frac{C_x}{C + C_x}\right)^2},$$

which clearly is maximum when $C_x = 0$, for which $(P_{av})_{max} = V_0^2/2R$.

(c) The energy stored in the variable capacitor is $U_{C_x} = \frac{1}{2} C_x V_C^2$, where V_C is the voltage on the two capacitors. Since V_C is of the form $V_C = V_{C0} \cos(\omega t + \phi)$, its average value is $(V_C)_{av} = V_{C0}/2$, and we have $(U_{C_x})_{av} = \frac{1}{4} C_x V_{C0}^2$. Using $V_{C0} = I_0 X_{C_{eq}} = I_0/\omega(C + C_x)$ and $\omega = 1/\sqrt{LC}$, the expression for the average energy stored in the variable capacitor is found as

$$(U_{C_x})_{av} = \frac{L C C_x I_0^2}{4(C + C_x)^2} \rightarrow (U_{C_x})_{av} = A \left(\frac{C_x}{L C_x^2 + C R^2 (C + C_x)^2} \right)$$

where A is independent of C_x . Maximum value of the average energy stored in the variable capacitor C_x is found as

$$\frac{d}{dC_x} (U_{C_x})_{av} = 0 \rightarrow C_x = \frac{RC}{\sqrt{L + R^2 C}}.$$

